Log Minimal Model Program for Kähler 3-folds

Omprokash Das
(Tata Institute of Fundamental Research, Mumbai)
(Joint work with Christopher Hacon)

23rd March 2023

Introduction

Introduction

- Minimal Model Program for projective varieties of dimension 3 over $\mathbb{C}$ was fully established in the late 80's and early 90's due the work of Ried, Mori, Miyaoka, Matsuki, Kawamata, Kollár, Shokurov and others.

Introduction

- Minimal Model Program for projective varieties of dimension 3 over $\mathbb{C}$ was fully established in the late 80's and early 90's due the work of Ried, Mori, Miyaoka, Matsuki, Kawamata, Kollár, Shokurov and others.
- There was also a major breakthrough in higher dimensions in 2006 due to Birkar, Cascini, Hacon and McKernan. The authors proved the existence of flip and the existence of minimal model for varieties of general type, for projective varieties over $\mathbb{C}$ of arbitrary dimension.

Introduction

- Minimal Model Program for projective varieties of dimension 3 over $\mathbb{C}$ was fully established in the late 80's and early 90's due the work of Ried, Mori, Miyaoka, Matsuki, Kawamata, Kollár, Shokurov and others.
- There was also a major breakthrough in higher dimensions in 2006 due to Birkar, Cascini, Hacon and McKernan. The authors proved the existence of flip and the existence of minimal model for varieties of general type, for projective varieties over $\mathbb{C}$ of arbitrary dimension.
- In the analytic category, one could ask a similar question: "Is it possible to develop a minimal model program for **compact Kähler manifolds**?"

Introduction (continued...)

Introduction (continued...)

- Let X be a compact Kähler manifold. **Mori's Bend and Break** fails on X.

Introduction (continued...)

- Let X be a compact Kähler manifold. **Mori's Bend and Break** fails on X.
- Another major obstacle: **Base-point free theorem**, which gives us the contraction of K_X-negative extremal rays for X projective.

Introduction (continued...)

- Let X be a compact Kähler manifold. **Mori's Bend and Break** fails on X.
- Another major obstacle: **Base-point free theorem**, which gives us the contraction of K_X-negative extremal rays for X projective. Let's recall the Base-point free theorem:

Introduction (continued...)

- Let X be a compact Kähler manifold. **Mori's Bend and Break** fails on X.
- Another major obstacle: **Base-point free theorem**, which gives us the contraction of K_X-negative extremal rays for X projective. Let's recall the Base-point free theorem:

Theorem

Let X be a smooth projective variety and D is a nef Cartier divisor. If $aD - K_X$ is nef and big, then mD is semi-ample for all $m \gg 0$, i.e. there is a contraction $f : X \to Y$ to a projective variety Y such that $mD = f^*H_Y$, where H_Y is an ample divisor on Y.

Introduction (continued...)

- Let X be a compact Kähler manifold. **Mori's Bend and Break** fails on X.
- Another major obstacle: **Base-point free theorem**, which gives us the contraction of K_X-negative extremal rays for X projective. Let's recall the Base-point free theorem:

Theorem

Let X be a smooth projective variety and D is a nef Cartier divisor. If $aD - K_X$ is nef and big, then mD is semi-ample for all $m \gg 0$, i.e. there is a contraction $f : X \to Y$ to a projective variety Y such that $mD = f^*H_Y$, where H_Y is an ample divisor on Y.

- If a Kähler manifold poses a big line bundle, then it is projective. So Base-point free theorem is not available for us.

Introduction (continued...)

Introduction (continued...)

- In fact, there are more troubles: The Mori cone $\overline{\mathrm{NE}}(X)$ could be too small to be useful when X is Kähler.

Introduction (continued...)

- In fact, there are more troubles: The Mori cone $\overline{\mathrm{NE}}(X)$ could be too small to be useful when X is Kähler.
- There are examples of compact Kähler manifolds X s.t. it doesn't have any positive dimensional subvariety, e.g. **simple tori**.

Introduction (continued...)

- In fact, there are more troubles: The Mori cone $\overline{\mathrm{NE}}(X)$ could be too small to be useful when X is Kähler.
- There are examples of compact Kähler manifolds X s.t. it doesn't have any positive dimensional subvariety, e.g. **simple tori**.
- For more discussion on these kind of examples, see: 'Compact Kähler 3-folds without non-trivial subvarieties.' by Campana, Demailly abd Verbitsky.

Introduction (continued...)

- In fact, there are more troubles: The Mori cone $\overline{\mathrm{NE}}(X)$ could be too small to be useful when X is Kähler.
- There are examples of compact Kähler manifolds X s.t. it doesn't have any positive dimensional subvariety, e.g. **simple tori**.
- For more discussion on these kind of examples, see: 'Compact Kähler 3-folds without non-trivial subvarieties.' by Campana, Demailly abd Verbitsky.
- So we need to enlarge the vectors spaces $\mathrm{NS}(X)_{\mathbb{R}}, N_1(X)$ as well as the cones $\mathrm{Nef}(X), \overline{\mathrm{NE}}(X)$, etc.

New Tools

New Tools

- Let X be a normal compact analytic variety.

New Tools

- Let X be a normal compact analytic variety.
- The **Bott-Chern** cohomology $\mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is defined as the d-closed $(1,1)$-forms with local potentials modulo $i\partial\bar{\partial}\varphi$, where φ is a smooth function on X.

New Tools

- Let X be a normal compact analytic variety.
- The **Bott-Chern** cohomology $\mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is defined as the d-closed $(1,1)$-forms with local potentials modulo $i\partial\bar{\partial}\varphi$, where φ is a smooth function on X.
- $\mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ can also be defined as the quotient space of d-closed $(1,1)$-currents with local potentials modulo $i\partial\bar{\partial}u$, where u is a **distribution** on X.

New Tools

- Let X be a normal compact analytic variety.
- The **Bott-Chern** cohomology $\mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is defined as the d-closed $(1,1)$-forms with local potentials modulo $i\partial\bar{\partial}\varphi$, where φ is a smooth function on X.
- $\mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ can also be defined as the quotient space of d-closed $(1,1)$-currents with local potentials modulo $i\partial\bar{\partial}u$, where u is a **distribution** on X.
- When X is a compact Kähler manifold, $\mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is the usual $H^{1,1}(X)$.
- We define $N^1(X) := \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$.

New Tools (Continued. . .)

New Tools (Continued...)

- Let X be a normal compact analytic variety.

New Tools (Continued. . .)

- Let X be a normal compact analytic variety.
- $N_1(X)$ is defined as the space of all real closed bi-dimension $(1,1)$ currents T module the equivalence relation: $T \equiv T' \Leftrightarrow T(\alpha) = T'(\alpha)$ for all real closed $(1,1)$ forms with α local potentials.

New Tools (Continued...)

- Let X be a normal compact analytic variety.
- $N_1(X)$ is defined as the space of all real closed bi-dimension $(1,1)$ currents T module the equivalence relation: $T \equiv T' \;\Leftrightarrow\; T(\alpha) = T'(\alpha)$ for all real closed $(1,1)$ forms with α local potentials.
- When X has rational singularities, $N^1(X) \times N_1(X) \to \mathbb{R}$ is a perfect pairing.

New Tools (Continued...)

- Let X be a normal compact analytic variety.
- $N_1(X)$ is defined as the space of all real closed bi-dimension $(1,1)$ currents T module the equivalence relation: $T \equiv T' \Leftrightarrow T(\alpha) = T'(\alpha)$ for all real closed $(1,1)$ forms with α local potentials.
- When X has rational singularities, $N^1(X) \times N_1(X) \to \mathbb{R}$ is a perfect pairing. In particular, $N^1(X)^* \cong N_1(X)$.

New Tools (Continued...)

- Let X be a normal compact analytic variety.
- $N_1(X)$ is defined as the space of all real closed bi-dimension $(1,1)$ currents T module the equivalence relation: $T \equiv T' \;\Leftrightarrow\; T(\alpha) = T'(\alpha)$ for all real closed $(1,1)$ forms with α local potentials.
- When X has rational singularities, $N^1(X) \times N_1(X) \to \mathbb{R}$ is a perfect pairing. In particular, $N^1(X)^* \cong N_1(X)$.
- When X is a compact Kähler manifold, $N_1(X) \cong H^{n-1,n-1}(X)$.

New Phenomenon

New Phenomenon

- In general, $\mathsf{NS}(X) \subseteq \mathsf{H}^{1,1}_{\mathrm{BC}}(X)$.

New Phenomenon

- In general, $\mathrm{NS}(X) \subseteq \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$. This inclusion could be strict even when X is projective.

New Phenomenon

- In general, $\mathrm{NS}(X) \subseteq \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$. This inclusion could be strict even when X is projective.
- For example, for any smooth projective $K3$ surface X, $h^{1,1}(X) = 20$, but there are $K3$ surfaces with Picard number smaller than 20.

New Phenomenon

- In general, $\mathrm{NS}(X) \subseteq \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$. This inclusion could be strict even when X is projective.
- For example, for any smooth projective $K3$ surface X, $h^{1,1}(X) = 20$, but there are $K3$ surfaces with Picard number smaller than 20.

Kähler-Mori Cone

Kähler-Mori Cone

- We define the Kähler-Mori cone $\overline{\mathrm{NA}}(X) \subseteq N_1(X)$ to be the closed cone generated by the classes of positive closed currents.

Kähler-Mori Cone

- We define the Kähler-Mori cone $\overline{\mathrm{NA}}(X) \subseteq N_1(X)$ to be the closed cone generated by the classes of positive closed currents. Note that for any curve $C \subseteq X$, the associated **currents of integration** T_C defined as $T_C(\eta) := \int_C \eta$ for all closed $(1,1)$ forms η, is closed positive bi-dimension $(1,1)$ currents. Thus $\overline{\mathrm{NE}}(X) \subseteq \overline{\mathrm{NA}}(X)$.

New Tools (Continued. . .)

New Tools (Continued. . .)

- An analytic variety X is called Kähler if there exists a Kähler form ω, i.e., a positive closed real $(1,1)$ form $\omega \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$ such that the following holds: for every point $x \in X$ there exists an open nbhd $x \in U \subseteq X$ and a closed embedding $\iota_U : U \hookrightarrow V$ into an open subset $V \subseteq \mathbb{C}^N$, and a strictly plurisubharmonic C^∞-function $f : V \to \mathbb{R}$ with $\omega|_{U \cap X_{\mathrm{sm}}} = (i\partial\bar{\partial} f)|_{U \cap X_{\mathrm{sm}}}$.

New Tools (Continued. . .)

- An analytic variety X is called Kähler if there exists a Kähler form ω, i.e., a positive closed real $(1,1)$ form $\omega \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$ such that the following holds: for every point $x \in X$ there exists an open nbhd $x \in U \subseteq X$ and a closed embedding $\iota_U : U \hookrightarrow V$ into an open subset $V \subseteq \mathbb{C}^N$, and a strictly plurisubharmonic C^∞-function $f : V \to \mathbb{R}$ with $\omega|_{U \cap X_{\mathrm{sm}}} = (i\partial\bar{\partial} f)|_{U \cap X_{\mathrm{sm}}}$.
- Let $u \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ be a class represented by a $(1,1)$ form α with local potentials. Then u is called **nef** if for some Kähler form ω on X and for every $\varepsilon > 0$ there exists $f_\varepsilon \in \mathcal{A}^0(X)$ such that $\alpha + i\partial\bar{\partial} f_\varepsilon \geq -\varepsilon\omega$.

New Tools (Continued...)

- An analytic variety X is called Kähler if there exists a Kähler form ω, i.e., a positive closed real $(1,1)$ form $\omega \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$ such that the following holds: for every point $x \in X$ there exists an open nbhd $x \in U \subseteq X$ and a closed embedding $\iota_U : U \hookrightarrow V$ into an open subset $V \subseteq \mathbb{C}^N$, and a strictly plurisubharmonic C^∞-function $f : V \to \mathbb{R}$ with $\omega|_{U \cap X_{\mathrm{sm}}} = (i\partial\bar{\partial}f)|_{U \cap X_{\mathrm{sm}}}$.
- Let $u \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ be a class represented by a $(1,1)$ form α with local potentials. Then u is called **nef** if for some Kähler form ω on X and for every $\varepsilon > 0$ there exists $f_\varepsilon \in \mathcal{A}^0(X)$ such that $\alpha + i\partial\bar{\partial}f_\varepsilon \geq -\varepsilon\omega$.
- Let $\mathcal{K} \subseteq N^1(X)$ is the open convex cone generated by the classes of Kähler forms, $\mathrm{Nef}(X) \subseteq \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is the closed of cone of nef classes.

New Tools (Continued...)

- An analytic variety X is called Kähler if there exists a Kähler form ω, i.e., a positive closed real $(1,1)$ form $\omega \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$ such that the following holds: for every point $x \in X$ there exists an open nbhd $x \in U \subseteq X$ and a closed embedding $\iota_U : U \hookrightarrow V$ into an open subset $V \subseteq \mathbb{C}^N$, and a strictly plurisubharmonic C^∞-function $f : V \to \mathbb{R}$ with $\omega|_{U \cap X_{\mathrm{sm}}} = (i\partial\bar{\partial} f)|_{U \cap X_{\mathrm{sm}}}$.
- Let $u \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ be a class represented by a $(1,1)$ form α with local potentials. Then u is called **nef** if for some Kähler form ω on X and for every $\varepsilon > 0$ there exists $f_\varepsilon \in \mathcal{A}^0(X)$ such that $\alpha + i\partial\bar{\partial} f_\varepsilon \geq -\varepsilon\omega$.
- Let $\mathcal{K} \subseteq N^1(X)$ is the open convex cone generated by the classes of Kähler forms, $\mathrm{Nef}(X) \subseteq \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is the closed of cone of nef classes. Then from a theorem of Demailly it follows that $\mathrm{Nef}(X) = \overline{\mathcal{K}}$.

New Tools (Continued. . .)

New Tools (Continued. . .)

- Let X be a normal compact Kähler variety.

New Tools (Continued...)

- Let X be a normal compact Kähler variety.
- $\mathrm{Nef}(X)$ is dual to $\overline{\mathrm{NA}}(X)$.

New Tools (Continued...)

- Let X be a normal compact Kähler variety.
- $\mathrm{Nef}(X)$ is dual to $\overline{\mathrm{NA}}(X)$.
- If $\alpha \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ such that $T(\alpha) > 0$ for all $T \in \overline{\mathrm{NA}}(X) \setminus \{0\}$, then α is a represented by a Kähler form.

Some bad new. . .

Some bad new. . .

- Let X be a compact Kähler manifold and L is a line bundle on X.

Some bad new…

- Let X be a compact Kähler manifold and L is a line bundle on X.
- If L is nef in the algebraic sense, i.e. $\int_C c_1(L) > 0$ for all curve $C \subseteq X$, it doesn't necessarily imply that $c_1(L) \in \mathrm{Nef}(X) = \overline{\mathcal{K}}$.

Some bad new. . .

- Let X be a compact Kähler manifold and L is a line bundle on X.
- If L is nef in the algebraic sense, i.e. $\int_C c_1(L) > 0$ for all curve $C \subseteq X$, it doesn't necessarily imply that $c_1(L) \in \mathrm{Nef}(X) = \overline{\mathcal{K}}$.
- Example: Let X be a smooth compact Kähler surface s.t. $a(X) := \text{ tr.deg.}_{\mathbb{C}}\mathbb{C}(X) = 1$. Then there is a $f : X \to C$ proper morphism such that all the curve in X are vertical over C.

Some bad new...

- Let X be a compact Kähler manifold and L is a line bundle on X.
- If L is nef in the algebraic sense, i.e. $\int_C c_1(L) > 0$ for all curve $C \subseteq X$, it doesn't necessarily imply that $c_1(L) \in \mathrm{Nef}(X) = \overline{\mathcal{K}}$.
- Example: Let X be a smooth compact Kähler surface s.t. $a(X) :=$ tr.deg.$_{\mathbb{C}}\mathbb{C}(X) = 1$. Then there is a $f : X \to C$ proper morphism such that all the curve in X are vertical over C.
- Let $p \in C$ and $D = -p \in \mathrm{NS}(C)$. Then $f^*D \cdot \Gamma \geq 0$ for all curves $\Gamma \subseteq X$ but f^*D is anti-effective, so $c_1(f^*D) \notin \mathrm{Nef}(X)$.

Escape from the disaster

Escape from the disaster

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.

Escape from the disaster

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.
- Assume that K_X is pseudo-effective. Then K_X **is algebraically nef if and only if it is analytically nef**.

Escape from the disaster

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.
- Assume that K_X is pseudo-effective. Then K_X **is algebraically nef if and only if it is analytically nef**.
- **Proof:** The if part is obvious. So assume that K_X is algebraically nef but not analytically nef. Boucksom-Zariski decomposition $K_X \equiv \sum a_i S_i + \beta$, where $a_i \geq 0$ and $\beta \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is **nef in codimension** 1, i.e. $\beta|_D$ is pseudo-effective for any prime Weil divisor D.

Escape from the disaster

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.
- Assume that K_X is pseudo-effective. Then K_X **is algebraically nef if and only if it is analytically nef**.
- **Proof:** The if part is obvious. So assume that K_X is algebraically nef but not analytically nef. Boucksom-Zariski decomposition $K_X \equiv \sum a_i S_i + \beta$, where $a_i \geq 0$ and $\beta \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is **nef in codimension** 1, i.e. $\beta|_D$ is pseudo-effective for any prime Weil divisor D.
- Since K_X and $K_X|_C$ is pseudo-effective for every curve $C \subseteq X$, by Păun's criteria, $K_X|_S$ is not pseudo-effective.

Escape from the disaster

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.
- Assume that K_X is pseudo-effective. Then K_X **is algebraically nef if and only if it is analytically nef**.
- **Proof:** The if part is obvious. So assume that K_X is algebraically nef but not analytically nef. Boucksom-Zariski decomposition $K_X \equiv \sum a_i S_i + \beta$, where $a_i \geq 0$ and $\beta \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is **nef in codimension** 1, i.e. $\beta|_D$ is pseudo-effective for any prime Weil divisor D.
- Since K_X and $K_X|_C$ is pseudo-effective for every curve $C \subseteq X$, by Păun's criteria, $K_X|_S$ is not pseudo-effective.
- Then from the decomposition $K_X \equiv \sum a_i S_i + \beta$ it follows that $S = S_i$ for some i.

Escape from the disaster

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.
- Assume that K_X is pseudo-effective. Then K_X **is algebraically nef if and only if it is analytically nef**.
- **Proof:** The if part is obvious. So assume that K_X is algebraically nef but not analytically nef. Boucksom-Zariski decomposition $K_X \equiv \sum a_i S_i + \beta$, where $a_i \geq 0$ and $\beta \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is **nef in codimension** 1, i.e. $\beta|_D$ is pseudo-effective for any prime Weil divisor D.
- Since K_X and $K_X|_C$ is pseudo-effective for every curve $C \subseteq X$, by Păun's criteria, $K_X|_S$ is not pseudo-effective.
- Then from the decomposition $K_X \equiv \sum a_i S_i + \beta$ it follows that $S = S_i$ for some i. From adjunction it follows that K_S is not pseudo-effective, so S is Moishezon.

Escape from the disaster

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.
- Assume that K_X is pseudo-effective. Then K_X **is algebraically nef if and only if it is analytically nef**.
- **Proof:** The if part is obvious. So assume that K_X is algebraically nef but not analytically nef. Boucksom-Zariski decomposition $K_X \equiv \sum a_i S_i + \beta$, where $a_i \geq 0$ and $\beta \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is **nef in codimension** 1, i.e. $\beta|_D$ is pseudo-effective for any prime Weil divisor D.
- Since K_X and $K_X|_C$ is pseudo-effective for every curve $C \subseteq X$, by Păun's criteria, $K_X|_S$ is not pseudo-effective.
- Then from the decomposition $K_X \equiv \sum a_i S_i + \beta$ it follows that $S = S_i$ for some i. From adjunction it follows that K_S is not pseudo-effective, so S is Moishezon.
- $\{C_t\} \subseteq S$ covering family. Then $K_X \cdot C_t = (K_X|_S) \cdot C_t < 0$, since $K_X|_S$ is not pseudo-effective.

Escape from the disaster

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.
- Assume that K_X is pseudo-effective. Then K_X **is algebraically nef if and only if it is analytically nef**.
- **Proof:** The if part is obvious. So assume that K_X is algebraically nef but not analytically nef. Boucksom-Zariski decomposition $K_X \equiv \sum a_i S_i + \beta$, where $a_i \geq 0$ and $\beta \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ is **nef in codimension** 1, i.e. $\beta|_D$ is pseudo-effective for any prime Weil divisor D.
- Since K_X and $K_X|_C$ is pseudo-effective for every curve $C \subseteq X$, by Păun's criteria, $K_X|_S$ is not pseudo-effective.
- Then from the decomposition $K_X \equiv \sum a_i S_i + \beta$ it follows that $S = S_i$ for some i. From adjunction it follows that K_S is not pseudo-effective, so S is Moishezon.
- $\{C_t\} \subseteq S$ covering family. Then $K_X \cdot C_t = (K_X|_S) \cdot C_t < 0$, since $K_X|_S$ is not pseudo-effective.This is a contradiction.

Cone Theorem in dim 3 [HP16]

Cone Theorem in dim 3 [HP16]

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.

Cone Theorem in dim 3 [HP16]

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.
- If K_X is pseudo-effective, but not nef, then there is a countable family of rational curves $\{C_i\}_{i \in I}$ such that $0 < -K_X \cdot C_i \leq 6$ and

Cone Theorem in dim 3 [HP16]

- Let X be a normal compact Kähler 3-fold with $\mathbb{Q}$-factorial terminal singularities.
- If K_X is pseudo-effective, but not nef, then there is a countable family of rational curves $\{C_i\}_{i \in I}$ such that $0 < -K_X \cdot C_i \leq 6$ and

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ \cdot [C_i].$$

- When K_X is not pseudo-effective, the cone decomposition looks a bit different than above.

Existence of MMP

Existence of MMP

Theorem (Höring and Perternell, 2015-2016)

Let X is be $\mathbb{Q}$-factorial compact Kähler 3-fold with terminal singularities. If K_X is pseudo-effective, then there is a finite sequence of K_X-flips and divisorial contractions:
$\phi : X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$ *such that* K_{X_n} *is nef.*

Existence of MMP

Theorem (Höring and Perternell, 2015-2016)

Let X is be $\mathbb{Q}$-factorial compact Kähler 3-fold with terminal singularities. If K_X is pseudo-effective, then there is a finite sequence of K_X-flips and divisorial contractions:
$\phi : X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$ *such that K_{X_n} is nef.*

Theorem (Höring and Perternell, 2015-16)

Let X be a $\mathbb{Q}$-factorial compact Kähler 3-fold with terminal singularities. If K_X is not pseudo-effective, then there is a finite sequence of K_X-flips and divisorial contractions:
$\phi : X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$ *and a fibration $f : X_n \to Z$ (called Mori fiber space) such that $-K_{X_n}$ is f-ample and the relative Picard number $\rho(X_n/Z) = 1$.*

Existence of Log MMP

Theorem (D- and Hacon, 2020)

Let (X, Δ) be a dlt pair, where X is a $\mathbb{Q}$-factorial compact Kähler 3-fold. If $K_X + \Delta$ is pseudo-effective, then there exists a finite sequence of $(K_X + \Delta)$-flips and divisorial contractions:
$\phi : X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$ *such that* $K_{X_n} + \phi_*\Delta$ *is nef.*

Existence of Log MMP

Theorem (D- and Hacon, 2020)

Let (X, Δ) be a dlt pair, where X is a $\mathbb{Q}$-factorial compact Kähler 3-fold. If $K_X + \Delta$ is pseudo-effective, then there exists a finite sequence of $(K_X + \Delta)$-flips and divisorial contractions: $\phi : X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$ such that $K_{X_n} + \phi_*\Delta$ is nef.

Theorem (D- and Hacon, 2020)

Let (X, Δ) be a dlt pair, where X is a $\mathbb{Q}$-factorial compact Kähler 3-fold. If $K_X + \Delta$ is not pseudo-effective, then there exists a finite a sequence of $(K_X + \Delta)$-flips and divisorial contractions $\phi : X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$ and a fibration $f : X_n \to Z$ such that $-(K_{X_n} + \phi_*\Delta)$ is f-ample and $\rho(X_n/Z) = 1$.

Existence and Termination of Flips

Existence and Termination of Flips

- If $f : X \to Z$ is a flipping contraction, then we get the existence of flip $f^+ : X^+ \to Z$ for FREE!

Existence and Termination of Flips

- If $f : X \to Z$ is a flipping contraction, then we get the existence of flip $f^+ : X^+ \to Z$ for FREE!
- If X has terminal singularity, then the existence of f^+ directly follows from Mori's proof, since his proof is analytic.

Existence and Termination of Flips

- If $f : X \to Z$ is a flipping contraction, then we get the existence of flip $f^+ : X^+ \to Z$ for FREE!
- If X has terminal singularity, then the existence of f^+ directly follows from Mori's proof, since his proof is analytic.
- If (X, Δ) is a log canonical pair and f is a $(K_X + \Delta)$-flipping contraction, then the existence of f^+ is due to Shokurov, because his proof is also analytic.

Existence and Termination of Flips

- If $f : X \to Z$ is a flipping contraction, then we get the existence of flip $f^+ : X^+ \to Z$ for FREE!
- If X has terminal singularity, then the existence of f^+ directly follows from Mori's proof, since his proof is analytic.
- If (X, Δ) is a log canonical pair and f is a $(K_X + \Delta)$-flipping contraction, then the existence of f^+ is due to Shokurov, because his proof is also analytic.
- Termination of flips is also analytic proof, that works too!

Existence and Termination of Flips

- If $f : X \to Z$ is a flipping contraction, then we get the existence of flip $f^+ : X^+ \to Z$ for FREE!
- If X has terminal singularity, then the existence of f^+ directly follows from Mori's proof, since his proof is analytic.
- If (X, Δ) is a log canonical pair and f is a $(K_X + \Delta)$-flipping contraction, then the existence of f^+ is due to Shokurov, because his proof is also analytic.
- Termination of flips is also analytic proof, that works too!
- So the main difficulty for us is the existence of contractions of negative extremal rays.

Base-point free conjecture on Kähler Variety

Base-point free conjecture on Kähler Variety

In the analytic category, there is a Base-point free conjecture which mimics the statement of the Base-point free theorem in the projective case with divisors replaced by cohomology classes.

Base-point free conjecture on Kähler Variety

In the analytic category, there is a Base-point free conjecture which mimics the statement of the Base-point free theorem in the projective case with divisors replaced by cohomology classes.

Conjecture

Let $(X, \Delta \geq 0)$ *be a klt pair, where* X *is a compact Kähler variety. Let* $\alpha \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ *be a nef class such that* $\alpha - (K_X + \Delta)$ *is nef and big. Then there is a proper morphism with connected fiber* $f : X \to Z$ *to a compact Kähler variety* Z *with rational singularity and* $\alpha = f^*\omega_Z$*, where* ω_Z *is a Kähler class on* Z*.*

Base-point free conjecture on Kähler Variety

In the analytic category, there is a Base-point free conjecture which mimics the statement of the Base-point free theorem in the projective case with divisors replaced by cohomology classes.

Conjecture

Let $(X, \Delta \geq 0)$ *be a klt pair, where* X *is a compact Kähler variety. Let* $\alpha \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ *be a nef class such that* $\alpha - (K_X + \Delta)$ *is nef and big. Then there is a proper morphism with connected fiber* $f : X \to Z$ *to a compact Kähler variety* Z *with rational singularity and* $\alpha = f^*\omega_Z$*, where* ω_Z *is a Kähler class on* Z*.*

We show that this conjecture holds in dimension 3.

Base-point free conjecture on Kähler Variety

In the analytic category, there is a Base-point free conjecture which mimics the statement of the Base-point free theorem in the projective case with divisors replaced by cohomology classes.

Conjecture

Let $(X, \Delta \geq 0)$ be a klt pair, where X is a compact Kähler variety. Let $\alpha \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ be a nef class such that $\alpha - (K_X + \Delta)$ is nef and big. Then there is a proper morphism with connected fiber $f : X \to Z$ to a compact Kähler variety Z with rational singularity and $\alpha = f^*\omega_Z$, where ω_Z is a Kähler class on Z.

We show that this conjecture holds in dimension 3.

Theorem (D- and Hacon)

Let $(X, \Delta \geq 0)$ be a klt pair, where X is a compact Kähler 3-fold. Let $\alpha \in \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ be a nef class such that $\alpha - (K_X + \Delta)$ is nef and big. Then there is a proper morphism with connected fibers $f : X \to Z$ to a compact Kähler variety Z with rational singularity and $\alpha = f^*\omega_Z$, where ω_Z is a Kähler class on Z.

Base-point free conjecture

- For $\Delta = 0$, X terminal singularity and $\alpha - K_X$ a **Kähler class**, this theorem was proved earlier by Tosatti and Zhang [TZ18] (when α nef but not big) and Höring (when α is nef and big).

Base-point free conjecture

- For $\Delta = 0$, X terminal singularity and $\alpha - K_X$ a **Kähler class**, this theorem was proved earlier by Tosatti and Zhang [TZ18] (when α nef but not big) and Höring (when α is nef and big).
- Unfortunately, this theorem is proved in our paper as an application of the Log MMP, in particular, it can not be used to prove the contractions of $(K_X + \Delta)$-negative extremal rays of $\overline{\mathrm{NA}}(X)$.

Blowing Down Theorem in Analytic Geometry

Blowing Down Theorem in Analytic Geometry

Theorem (Fujiki 1974)

Let X be normal compact analytic variety and S a $\mathbb{Q}$-Cartier prime Weil divisor on X with Cartier index $m > 0$. Let $g : S \to B$ be a contraction and $\mathcal{O}_S(-mS)$ is f-ample. Then there is normal compact analytic variety Y containing B and a bimeromorphic map $f : X \to Y$ such that $f|_S = g$ and $f|_{X \setminus S}$ is isomorphic to $Y \setminus B$

Contraction of K_X-negative extremal rays (Höring and Peternell, 2015-16)

Contraction of K_X-negative extremal rays (Höring and Peternell, 2015-16)

- Let X be a $\mathbb{Q}$-factorial compact Kähler 3-fold with terminal singularities. Assume that K_X is pseudo-effective and let R be a K_X-negative extremal ray of $\overline{\mathrm{NA}}(X)$.

Contraction of K_X-negative extremal rays (Höring and Peternell, 2015-16)

- Let X be a $\mathbb{Q}$-factorial compact Kähler 3-fold with terminal singularities. Assume that K_X is pseudo-effective and let R be a K_X-negative extremal ray of $\overline{\mathsf{NA}}(X)$.
- It can be shown (as in the projective case) that there exists a nef class $\alpha \in N^1(X) = \mathsf{H}^{1,1}_{\mathrm{BC}}(X)$ such that

$$\alpha^{\perp} \cap \overline{\mathsf{NA}}(X) := \{\gamma \in \overline{\mathsf{NA}}(X) \mid \alpha \cdot \gamma = 0\} = R.$$

Contraction of K_X-negative extremal rays (Höring and Peternell, 2015-16)

- Let X be a $\mathbb{Q}$-factorial compact Kähler 3-fold with terminal singularities. Assume that K_X is pseudo-effective and let R be a K_X-negative extremal ray of $\overline{\mathsf{NA}}(X)$.
- It can be shown (as in the projective case) that there exists a nef class $\alpha \in N^1(X) = \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ such that

$$\alpha^\perp \cap \overline{\mathsf{NA}}(X) := \{\gamma \in \overline{\mathsf{NA}}(X) \mid \alpha \cdot \gamma = 0\} = R.$$

- Up to a rescaling of α it follows that $\alpha - K_X$ is a Kähler class, say $\alpha = K_X + \omega$, where ω is a Kähler class on X.

Contraction of K_X-negative extremal rays (Höring and Peternell, 2015-16)

- Let X be a $\mathbb{Q}$-factorial compact Kähler 3-fold with terminal singularities. Assume that K_X is pseudo-effective and let R be a K_X-negative extremal ray of $\overline{\mathrm{NA}}(X)$.
- It can be shown (as in the projective case) that there exists a nef class $\alpha \in N^1(X) = \mathrm{H}^{1,1}_{\mathrm{BC}}(X)$ such that

$$\alpha^{\perp} \cap \overline{\mathrm{NA}}(X) := \{\gamma \in \overline{\mathrm{NA}}(X) \mid \alpha \cdot \gamma = 0\} = R.$$

- Up to a rescaling of α it follows that $\alpha - K_X$ is a Kähler class, say $\alpha = K_X + \omega$, where ω is a Kähler class on X.
- Note that $\alpha = K_X + \omega$ is big, and hence $\alpha^3 > 0$, since it is a sum of a pseudo-effective class and a Kähler class.

Contraction of K_X-negative extremal rays (continued...)

- The null locus

$$\mathrm{Null}(\alpha) := \bigcup_{V \subseteq X,\ \dim V > 0,\ \alpha^{\dim V} \cdot V = 0} V \subsetneq X$$

is a countable union of proper subvarieties of X.

Contraction of K_X-negative extremal rays (continued...)

- The null locus

$$\mathrm{Null}(\alpha) := \bigcup_{V \subseteq X,\ \dim V > 0,\ \alpha^{\dim V} \cdot V = 0} V \subsetneq X$$

  is a countable union of proper subvarieties of X.
- By a theorem of Collins and Tosatti [CT15], $\mathrm{Null}(\alpha)$ is a closed analytic subset of X.

Contraction of K_X-negative extremal rays (continued...)

- The null locus

$$\mathrm{Null}(\alpha) := \bigcup_{V \subseteq X,\ \dim V > 0,\ \alpha^{\dim V} \cdot V = 0} V \subsetneq X$$

  is a countable union of proper subvarieties of X.
- By a theorem of Collins and Tosatti [CT15], $\mathrm{Null}(\alpha)$ is a closed analytic subset of X.
- With some work it can be shown that $\mathrm{Null}(\alpha)$ is exactly the locus covered by the curves in the extremal ray R.

Contraction of K_X-negative extremal rays (continued...)

- The null locus

$$\mathrm{Null}(\alpha) := \bigcup_{V \subseteq X,\ \dim V > 0,\ \alpha^{\dim V} \cdot V = 0} V \subsetneq X$$

  is a countable union of proper subvarieties of X.
- By a theorem of Collins and Tosatti [CT15], $\mathrm{Null}(\alpha)$ is a closed analytic subset of X.
- With some work it can be shown that $\mathrm{Null}(\alpha)$ is exactly the locus covered by the curves in the extremal ray R.
- Now there are three cases:

Contraction of K_X-negative extremal rays (continued...)

- The null locus

$$\mathrm{Null}(\alpha) := \bigcup_{V \subseteq X,\ \dim V > 0,\ \alpha^{\dim V} \cdot V = 0} V \subsetneq X$$

  is a countable union of proper subvarieties of X.
- By a theorem of Collins and Tosatti [CT15], $\mathrm{Null}(\alpha)$ is a closed analytic subset of X.
- With some work it can be shown that $\mathrm{Null}(\alpha)$ is exactly the locus covered by the curves in the extremal ray R.
- Now there are three cases:
  1. $\mathrm{Null}(\alpha)$ is a finite union of curves.

Contraction of K_X-negative extremal rays (continued...)

- The null locus

$$\mathrm{Null}(\alpha) := \bigcup_{V \subseteq X,\ \dim V > 0,\ \alpha^{\dim V} \cdot V = 0} V \subsetneq X$$

  is a countable union of proper subvarieties of X.
- By a theorem of Collins and Tosatti [CT15], $\mathrm{Null}(\alpha)$ is a closed analytic subset of X.
- With some work it can be shown that $\mathrm{Null}(\alpha)$ is exactly the locus covered by the curves in the extremal ray R.
- Now there are three cases:
  1. $\mathrm{Null}(\alpha)$ is a finite union of curves.
  2. $\mathrm{Null}(\alpha)$ is an irreducible surface S and $\alpha|_S \equiv 0$.

Contraction of K_X-negative extremal rays (continued...)

- The null locus

$$\mathrm{Null}(\alpha) := \bigcup_{V \subseteq X,\ \dim V > 0,\ \alpha^{\dim V} \cdot V = 0} V \subsetneq X$$

  is a countable union of proper subvarieties of X.

- By a theorem of Collins and Tosatti [CT15], $\mathrm{Null}(\alpha)$ is a closed analytic subset of X.
- With some work it can be shown that $\mathrm{Null}(\alpha)$ is exactly the locus covered by the curves in the extremal ray R.
- Now there are three cases:
  1. $\mathrm{Null}(\alpha)$ is a finite union of curves.
  2. $\mathrm{Null}(\alpha)$ is an irreducible surface S and $\alpha|_S \equiv 0$.
  3. $\mathrm{Null}(\alpha)$ is an irreducible surface S and $\alpha|_S \not\equiv 0$.

Filliping contraction (Sketch of the Proof)

Filliping contraction (Sketch of the Proof)

Case I: Assume that $Z = \mathrm{Null}(\alpha)$ is a finite union of curves.

Filliping contraction (Sketch of the Proof)

Case I: Assume that $Z = \text{Null}(\alpha)$ is a finite union of curves.

- In this case the morphism g is simply the map that sends the connected component of Z to different points.

Filliping contraction (Sketch of the Proof)

Case I: Assume that $Z = \mathrm{Null}(\alpha)$ is a finite union of curves.

- In this case the morphism g is simply the map that sends the connected component of Z to different points.
- The difficult part here is to check that the conormal sheaf of $Z = \mathrm{Null}(\alpha)$ restricted to the fibers of g is ample.

Filliping contraction (Sketch of the Proof)

Case I: Assume that $Z = \mathrm{Null}(\alpha)$ is a finite union of curves.

- In this case the morphism g is simply the map that sends the connected component of Z to different points.
- The difficult part here is to check that the conormal sheaf of $Z = \mathrm{Null}(\alpha)$ restricted to the fibers of g is ample.
- By a theorem of Boucksom [Bou04] and Collins and Tosatti [CT15] there exists a projective bimeromorphic morphism $\mu : X' \to X$ from a Kähler manifold X' and a Kähler form ω' on X' such that $\mathrm{Null}(\alpha) = \mu(\mathrm{Ex}(\mu))$ and

$$\mu^*\alpha = \omega' + E,$$

where $E \geq 0$ is an effective divisor s.t. $\mathrm{Supp}(E) = \mathrm{Ex}(\mu)$.

Filliping contraction (Sketch of the Proof)

Case I: Assume that $Z = \mathrm{Null}(\alpha)$ is a finite union of curves.

- In this case the morphism g is simply the map that sends the connected component of Z to different points.
- The difficult part here is to check that the conormal sheaf of $Z = \mathrm{Null}(\alpha)$ restricted to the fibers of g is ample.
- By a theorem of Boucksom [Bou04] and Collins and Tosatti [CT15] there exists a projective bimeromorphic morphism $\mu : X' \to X$ from a Kähler manifold X' and a Kähler form ω' on X' such that $\mathrm{Null}(\alpha) = \mu(\mathrm{Ex}(\mu))$ and

$$\mu^*\alpha = \omega' + E,$$

  where $E \geq 0$ is an effective divisor s.t. $\mathrm{Supp}(E) = \mathrm{Ex}(\mu)$.
- Since for any curve $C \subseteq \mathrm{Null}(\alpha)$, $\alpha|_C \equiv 0$, it follows that $-E|_E \equiv \omega|_E$, i.e., the conormal sheaf of E is (globally) an ample divisor on E.

Flipping and Divisorial contractions (Sketch of the Proof)

- Thus by the blowing down theorem, there is a proper bimeromorphic morphism $\pi : X' \to Y$ which contracts the connected component of E to points. Then by the Rigidity lemma, there is a bimeromorphic morphism $f : X \to Y$ such that π factorizes through it. In particular, f contracts the connected components of $\mathrm{Null}(\alpha)$. This is the **flipping contraction**.

Flipping and Divisorial contractions (Sketch of the Proof)

- Thus by the blowing down theorem, there is a proper bimeromorphic morphism $\pi : X' \to Y$ which contracts the connected component of E to points. Then by the Rigidity lemma, there is a bimeromorphic morphism $f : X \to Y$ such that π factorizes through it. In particular, f contracts the connected components of $\mathrm{Null}(\alpha)$. This is the **flipping contraction**.

Case II: $\mathrm{Null}(\alpha) = S$ and $\alpha|_S \equiv 0$.

Flipping and Divisorial contractions (Sketch of the Proof)

- Thus by the blowing down theorem, there is a proper bimeromorphic morphism $\pi : X' \to Y$ which contracts the connected component of E to points. Then by the Rigidity lemma, there is a bimeromorphic morphism $f : X \to Y$ such that π factorizes through it. In particular, f contracts the connected components of $\mathrm{Null}(\alpha)$. This is the **flipping contraction**.

Case II: $\mathrm{Null}(\alpha) = S$ and $\alpha|_S \equiv 0$.

- In this case $S \cdot C < 0$ for all curves $C \subseteq$ such that $[C] \in R$. So with this one can show that $\alpha - \varepsilon S$ is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$ for some $\varepsilon > 0$.

Flipping and Divisorial contractions (Sketch of the Proof)

- Thus by the blowing down theorem, there is a proper bimeromorphic morphism $\pi : X' \to Y$ which contracts the connected component of E to points. Then by the Rigidity lemma, there is a bimeromorphic morphism $f : X \to Y$ such that π factorizes through it. In particular, f contracts the connected components of $\mathrm{Null}(\alpha)$. This is the **flipping contraction**.

Case II: $\mathrm{Null}(\alpha) = S$ and $\alpha|_S \equiv 0$.

- In this case $S \cdot C < 0$ for all curves $C \subseteq$ such that $[C] \in R$. So with this one can show that $\alpha - \varepsilon S$ is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$ for some $\varepsilon > 0$.
- Thus $\alpha = \varepsilon S + \omega$, where ω is a Kähler class on X. In particular, $-S|_S \equiv \omega|_S$ is an ample divisor on S.

Flipping and Divisorial contractions (Sketch of the Proof)

- Thus by the blowing down theorem, there is a proper bimeromorphic morphism $\pi : X' \to Y$ which contracts the connected component of E to points. Then by the Rigidity lemma, there is a bimeromorphic morphism $f : X \to Y$ such that π factorizes through it. In particular, f contracts the connected components of $\mathrm{Null}(\alpha)$. This is the **flipping contraction**.

Case II: $\mathrm{Null}(\alpha) = S$ and $\alpha|_S \equiv 0$.

- In this case $S \cdot C < 0$ for all curves $C \subseteq$ such that $[C] \in R$. So with this one can show that $\alpha - \varepsilon S$ is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$ for some $\varepsilon > 0$.
- Thus $\alpha = \varepsilon S + \omega$, where ω is a Kähler class on X. In particular, $-S|_S \equiv \omega|_S$ is an ample divisor on S.
- Therefore by the Bllowing down theorem there is a projective birmeromorphic $f : X \to Y$ such that $f(S) = \mathrm{pt}$.

Divisorial Contraction (continued...)

Divisorial Contraction (continued...)

Case III: Null(α) $= S$ and $\alpha|_S \not\equiv 0$;

Divisorial Contraction (continued...)

Case III: Null(α) $= S$ and $\alpha|_S \not\equiv 0$; (this is the hardest case!)

Divisorial Contraction (continued...)

Case III: Null(α) $= S$ and $\alpha|_S \not\equiv 0$; (this is the hardest case!)

- In this case two arbitrary points of S con not be connected by a chain of curves which are all α-trivial. In particular, the nef dimension of $\alpha|_S$ is 1.

Divisorial Contraction (continued...)

Case III: Null$(\alpha) = S$ and $\alpha|_S \not\equiv 0$; (this is the hardest case!)

- In this case two arbitrary points of S con not be connected by a chain of curves which are all α-trivial. In particular, the nef dimension of $\alpha|_S$ is 1.
- But in order to use the nef reduction map and nef dimension we need to go to the normalization of S, say $\nu : \tilde{S} \to S$.

Divisorial Contraction (continued...)

Case III: Null$(\alpha) = S$ and $\alpha|_S \not\equiv 0$; (this is the hardest case!)

- In this case two arbitrary points of S con not be connected by a chain of curves which are all α-trivial. In particular, the nef dimension of $\alpha|_S$ is 1.
- But in order to use the nef reduction map and nef dimension we need to go to the normalization of S, say $\nu : \tilde{S} \to S$.
- Then the nef dimension $n(\nu^*(\alpha|_{\tilde{S}})) = 1$. Let $\tilde{g} : \tilde{S} \to B$ be the nef reduction map.

Divisorial Contraction (continued...)

Case III: $\text{Null}(\alpha) = S$ and $\alpha|_S \not\equiv 0$; (this is the hardest case!)

- In this case two arbitrary points of S con not be connected by a chain of curves which are all α-trivial. In particular, the nef dimension of $\alpha|_S$ is 1.
- But in order to use the nef reduction map and nef dimension we need to go to the normalization of S, say $\nu : \tilde{S} \to S$.
- Then the nef dimension $n(\nu^*(\alpha|_{\tilde{S}})) = 1$. Let $\tilde{g} : \tilde{S} \to B$ be the nef reduction map.
- We want to show that this morphism $\tilde{g}$ descends to a morphism $g : S \to B$. This turns out to be a surprisingly hard problem!

Divisorial Contraction (continued...)

Case III: $\mathrm{Null}(\alpha) = S$ and $\alpha|_S \not\equiv 0$; (this is the hardest case!)

- In this case two arbitrary points of S con not be connected by a chain of curves which are all α-trivial. In particular, the nef dimension of $\alpha|_S$ is 1.
- But in order to use the nef reduction map and nef dimension we need to go to the normalization of S, say $\nu : \tilde{S} \to S$.
- Then the nef dimension $n(\nu^*(\alpha|_{\tilde{S}})) = 1$. Let $\tilde{g} : \tilde{S} \to B$ be the nef reduction map.
- We want to show that this morphism $\tilde{g}$ descends to a morphism $g : S \to B$. This turns out to be a surprisingly hard problem!
- When X has terminal singularity, a computation of intersection number shows that S is smooth in a nbhd of the general fibers of $\tilde{g} : \tilde{S} \to B$.

Divisorial Contraction (continued...)

Case III: $\mathrm{Null}(\alpha) = S$ and $\alpha|_S \not\equiv 0$; (this is the hardest case!)

- In this case two arbitrary points of S con not be connected by a chain of curves which are all α-trivial. In particular, the nef dimension of $\alpha|_S$ is 1.
- But in order to use the nef reduction map and nef dimension we need to go to the normalization of S, say $\nu : \tilde{S} \to S$.
- Then the nef dimension $n(\nu^*(\alpha|_{\tilde{S}})) = 1$. Let $\tilde{g} : \tilde{S} \to B$ be the nef reduction map.
- We want to show that this morphism $\tilde{g}$ descends to a morphism $g : S \to B$. This turns out to be a surprisingly hard problem!
- When X has terminal singularity, a computation of intersection number shows that S is smooth in a nbhd of the general fibers of $\tilde{g} : \tilde{S} \to B$.
- An explicit computations then shows that $\tilde{g}$ descends to a morphism $g : S \to B$.

Divisorial Contraction (continued...)

Case III: Null$(\alpha) = S$ and $\alpha|_S \not\equiv 0$; (this is the hardest case!)

- In this case two arbitrary points of S con not be connected by a chain of curves which are all α-trivial. In particular, the nef dimension of $\alpha|_S$ is 1.
- But in order to use the nef reduction map and nef dimension we need to go to the normalization of S, say $\nu : \tilde{S} \to S$.
- Then the nef dimension $n(\nu^*(\alpha|_{\tilde{S}})) = 1$. Let $\tilde{g} : \tilde{S} \to B$ be the nef reduction map.
- We want to show that this morphism $\tilde{g}$ descends to a morphism $g : S \to B$. This turns out to be a surprisingly hard problem!
- When X has terminal singularity, a computation of intersection number shows that S is smooth in a nbhd of the general fibers of $\tilde{g} : \tilde{S} \to B$.
- An explicit computations then shows that $\tilde{g}$ descends to a morphism $g : S \to B$.
- So we are done again by the blowing down theorem.

Thank you!

S. Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*,
Ann. Sci. École Norm. Sup. (4) **37**(1), 45–76 (2004).

T. C. Collins and V. Tosatti, *Kähler currents and null loci*,
Invent. Math. **202**(3), 1167–1198 (2015).

A. Höring and T. Peternell, *Minimal models for Kähler threefolds*,
Invent. Math. **203**(1), 217–264 (2016).

V. Tosatti and Y. Zhang, *Finite time collapsing of the Kähler-Ricci flow on threefolds*,
Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **18**(1), 105–118 (2018).